SOME STEADY MOTIONS OF A GRAVITATING GYROSTAT AND SPHEROID AND THEIR STABILITY

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Some steady motions of two gravitating bodies, one of them a spheroid and the other a gyrostat, are considered. The cases of a dynamically asymmetric and dynamically symmetric gyrostat are investigated. Sufficient conditions of stability are derived for the case of a dynamically symmetric gyrostat.

1. Let us introduce the following notation: $O\xi_1\xi_2\xi_3$ is a stationary coordinate system (see Fig. 1), $G\eta_1\eta_2\eta_3$ is a Koenig coordinate system with its origin at the center of mass G of the spheroid + gyrostat system whose axes are parallel to those of the stationary coordinate system, $Cx_1x_2x_3$ is a moving coordinate system whose axes lie along the principal central axes of inertia of the gyrostat, $Py_1y_2y_3$ is a moving coordinate system

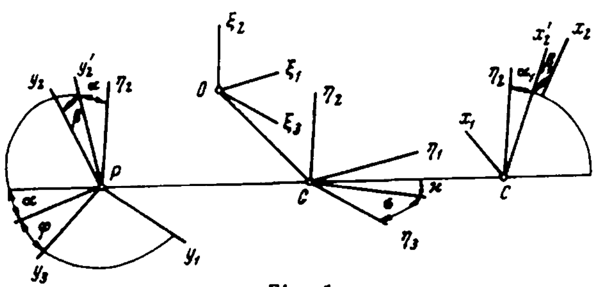

Fig. 1

whose axes lie along the principal central axes of inertia of the spheroid (the axis y_2 lies along the axis of dynamic symmetry of the spheroid), R_1, σ, $\varkappa$ and R_2, σ, $-\varkappa$ (where $R_1 + R_2 = R$) are the spherical coordinates of the centers of mass of the gyrostat and spheroid, respectively, relative to the Koenig coordinate system, σ is the longitude, $\varkappa$ is the latitude, α, β, φ are the Krylov angles, β is the angle of deviation of the dynamic symmetry axis y_2 of the spheroid from the plane Q passing through the line PC of the centers of mass and the axis η_2, α is the angle between the axis η_2 and the projection of the axis y_2 onto the plane Q, φ is the angle of proper rotation of the spheroid, α_1, β_1, φ_1 are the Krylov angles, where β_1 is the angle of deviation of the axis x_2 of the gyrostat from the plane Q, α_1 is the angle between the axis η_2 and the projection of the axis x_2 onto the plane Q, φ_1 is the angle between the axis x_3 and the line of intersection of the planes Q and Cx_1x_3, β_1, β_2, β_3 are the cosines of the angles between the axis η_2 and the axes x_1, x_2, x_3, respectively, γ_1, γ_2, γ_3 are the cosines of the angles between the radius vector $\mathbf{R}_1$ of the center of mass of the gyrostat with respect to the point G and the axes x_1, x_2, x_3, respectively, γ, γ', γ'' are the cosines of the angles between the radius vector $\mathbf{R}_2$ of the center of mass of the spheroid with respect to the point G and the axes y_1, y_2, y_3, respectively, f is the graviation constant, M_1, M_2; A_1, A_2, A_3; B_1, B_2, B_3 ($B_1 = B_3$) are masses and principal central moments of inertia of the gyrostat and spheroid, respectively, k_1, k_2, k_3 are the projections of the gyrostatic moment on the axes

x_1, x_2, x_3; $\omega_1, \omega_2, \omega_3$ are the projections of the absolute angular velocity of the gyrostat on the same axes, $\Omega_1, \Omega_2, \Omega_3$ are the projections of the absolute angular velocity of the spheroid on the axes y_1, y_2, y_3.

The kinetic energy T of the spheroid + gyrostat system is given by

$$T = {}^1/_2\, M_0 (R^{\cdot 2} + R\sigma^{\cdot 2} \cos^2 \varkappa + R^2 \varkappa^{\cdot 2}) + {}^1/_2 \Big\{ \sum_{i=1}^{3} A_i [\sigma^{\cdot} \beta_i + F_i (\varkappa^{\cdot}, \beta_j^{\cdot}, \gamma_j^{\cdot})]^2 +$$
$$+ \sum_{i=1}^{3} k_i [\sigma^{\cdot} \beta_i + F_i (\varkappa^{\cdot}, \beta_j^{\cdot}, \gamma_j^{\cdot})] + B_1 [\sigma^{\cdot 2} (\sin^2 \alpha + \cos^2 \alpha \sin^2 \beta) +$$
$$+ 2\sigma^{\cdot} \beta^{\cdot} \sin \alpha - 2\sigma^{\cdot} \alpha^{\cdot} \cos \alpha \sin \beta \cos \beta + \alpha^{\cdot 2} \cos^2 \beta + \beta^{\cdot 2}) + B_2 [\sigma^{\cdot 2} \cos^2 \alpha \cos^2 \beta +$$
$$+ 2\sigma^{\cdot} \varphi^{\cdot} \cos \alpha \cos \beta + 2\sigma^{\cdot} \alpha^{\cdot} \cos \alpha \cos \beta \sin \beta + \varphi^{\cdot 2} + 2\varphi^{\cdot} \alpha^{\cdot} \sin \beta + \alpha^{\cdot 2} \sin^2 \beta] \quad (1.1)$$
$$\left(M_0 = \frac{M_1 M_2}{M_1 + M_2} \right)$$

where the functions $F_i (\varkappa^{\cdot}, \beta_j^{\cdot}, \gamma_j^{\cdot})$ vanish for $\varkappa^{\cdot} = \beta_j^{\cdot} = \gamma_j^{\cdot} = 0$ $(i, j = 1, 2, 3)$.

The potential energy of the Newtonian attraction forces is given by the expression [1]

$$\Pi = \frac{3M_2 f}{2R^3} \left[A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2 - \frac{A_1 + A_2 + A_3}{3} \right] - f\, \frac{M_1 M_2}{R} +$$
$$+ \frac{3M_1 f}{2R^3} \Big\{ B_1 [\sin^2 (\alpha - \varkappa) \sin^2 \beta + \cos^2 (\alpha - \varkappa)] + \quad (1.2)$$
$$+ B_2 \sin^2 (\alpha - \varkappa) \cos^2 \beta - \frac{2B_1 + B_2}{3} \Big\}$$

2. The equations of motion of the spheroid + gyrostat system can be written in the form of Lagrange equations, where the Lagrangian coordinates q_i are the variables R, $\varkappa$, σ, α, α_1, β, β_1, φ, φ_1. These equations have the energy integral

$$T + \Pi = h = \text{const}$$

for the motion of the system relative to the Koenig axes.

Moreover, as we see from (1.1) and (1.2), the coordinates σ and φ are cyclical and correspond to the first integrals

$$\frac{\partial L}{\partial \sigma^{\cdot}} = M_0 R^2 \sigma^{\cdot} \cos^2 \varkappa + \sum_{i=1}^{3} \{A_i \beta_i [\sigma^{\cdot} \beta_i + F_i (\varkappa^{\cdot}, \beta_j^{\cdot}, \gamma_j^{\cdot})] + k_i \beta_i\} +$$
$$+ B_1 [\sigma^{\cdot} (\sin^2 \alpha + \cos^2 \alpha \sin^2 \beta) - \alpha^{\cdot} \cos \alpha \sin \beta \cos \beta + \beta^{\cdot} \sin \alpha] + \quad (2.1)$$
$$+ B_2 [\sigma^{\cdot} \cos^2 \alpha \cos^2 \beta + \varphi^{\cdot} \cos \alpha \cos \beta + \alpha^{\cdot} \cos \alpha \cos \beta \sin \beta) = K_\sigma$$

$$\frac{\partial L}{\partial \varphi^{\cdot}} = B_2 (\sigma^{\cdot} \cos \alpha \cos \beta + \varphi^{\cdot} + \alpha^{\cdot} \sin \beta) = K_\varphi \quad (2.2)$$

which express the constancy of the moment of momenta of the system (in its motion relative to the Koenig axes) with respect to the axis n_2 and the constancy of the moment of momenta of the spheroid (in its motion relative to the Koenig system) with respect to its proper axis of rotation y_3.

The second integral implies that the projection of the angular velocity of the spheroid on the axis y_2 is constant.

Ignoring the cyclical coordinates σ and φ, we construct the Routh function

$$R = L - \sigma^{\cdot} K_\sigma - \varphi^{\cdot} K_\varphi = R_2 + R_1 + R_0 \qquad (R_0 = - W)$$

Here R_i is a form of degree i in the generalized velocities $R^{\cdot}$, $\varkappa^{\cdot}$, $\alpha^{\cdot}$, $\alpha_1^{\cdot}$, $\beta^{\cdot}$, $\beta_1^{\cdot}$, $\varphi_1^{\cdot}$.

Making use of the relations

$$\beta_1^2+\beta_2^2+\beta_3^2=1, \qquad \gamma_1^2+\gamma_2^2+\gamma_3^2=1$$
$$\chi=\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3-\sin\varkappa=0$$

we can rewrite the altered potential energy of the spheroid + gyrostat system as

$$W(R,\varkappa,\alpha,\beta,\beta_1,\beta_3,\gamma_1,\gamma_2)=\frac{K^2}{2S}+\frac{K_\varphi^2}{2B_2}+\Pi$$

$$K=K_\sigma-K_\varphi\cos\alpha\cos\beta-k_1\beta_1-k_2(1-\beta_1^2-\beta_3^2)^{1/2}-k_3\beta_3$$
$$S=M_0R^2\cos^2\varkappa+B_1(\sin^2\alpha+\cos^2\alpha\sin^2\beta)+(A_1-A_2)\beta_1^2+(A_3-A_2)\beta_3^2+A_2$$

Introducing the Lagrange multiplier λ, we can determine the steady motions of our mechanical system from the equation $\delta W_1=0$ as the fixed points of the function $W_1=W+\lambda\varkappa$.

This equation has the following solutions (z_0 is the value of the function z in the corresponding steady motion):

$$R=R_0,\quad \varkappa=0,\quad \alpha=0,\quad \beta=0,\quad \beta_1=\sin\theta_0$$
$$\beta_3=0,\quad \gamma_1=0,\quad \gamma_2=0,\quad \lambda=0 \tag{2.3}$$
$$R=R_0,\quad \varkappa=0,\quad \alpha={}^1/_2\pi,\quad \beta=0,\quad K_\varphi=0,\quad \beta_1=\sin\theta_0$$
$$\beta_3=0,\quad \gamma_1=0,\quad \gamma_2=0,\quad \lambda=0 \tag{2.4}$$
$$R=R_0,\quad \varkappa=0,\quad \alpha=0,\quad \cos\beta=\cos\beta_0=K_\varphi/\omega_0B_1,\quad \beta_1=\sin\theta_0$$
$$\beta_3=0,\quad \gamma_1=0,\quad \gamma_2=0,\quad \lambda=0 \tag{2.5}$$
$$R=R_0,\ \varkappa=\varkappa_0,\ \cos\alpha=\cos\alpha_0=\frac{\omega_0K_\varphi}{\omega_0^2B_1+3fM_1R^{-3}(B_1-B_2)}+\delta \tag{2.6}$$
$$\beta=0,\quad \beta_1=0,\quad \beta_3=\sin(\theta_0+\varkappa_0),\quad \gamma_1=0,\ \gamma_2=-\sin\theta_0$$
$$\lambda=\frac{3fM_1}{2R^3}(A_2-A_3)\sin\theta_0$$

Here $\varkappa_0$ and δ are quantities of the order of l^2/R^2 (l is the characteristic dimension of the smaller body).

We note that if

$$M_2(A_2-A_3)\sin^2\theta_0=M_1(B_1-B_2)\sin^2\alpha_0$$

then $\varkappa_0=\delta=0$ in solution (2.6).

Solutions (2.3)–(2.5) exist under the conditions

$$M_0\omega_0^2R_0^3=f\{M_1M_2-{}^3/_2R_0^{-2}[(2A_3-A_1-A_2)M_2+(B_1-B_2)M_1]\}$$
$$k_3=0,\quad k_2\sin\theta_0-k_1\cos\theta_0={}^1/_2(A_1-A_2)\omega_0\sin^2\theta_0$$

Solution (2.6) exists under the conditions

$$M_0\omega_0^2R_0^2\cos^2\varkappa_0=f\{M_1M_2-{}^3/_2M_2R_0^{-2}[(A_2-A_3)\sin^2\theta_0+{}^1/_3(2A_3-A_1-A_2)]-$$
$$-{}^3/_2M_1R_0^{-2}[B_1\cos^2(\alpha_0-\varkappa_0)+B_2\sin^2(\alpha_0-\varkappa_0)-{}^1/_3(2B_1+B_2)]\}$$

$$k_1=0,\ \omega_0[k_2\sin(\theta_0+\varkappa_0)-k_3\cos(\theta_0+\varkappa_0)]+{}^1/_2(A_2-A_3)\omega_0^2\times$$
$$\times\sin^2(\theta_0+\varkappa_0)+{}^3/_2fM_1R^{-3}(A_2-A_3)\sin2\theta_0=0$$

$$M_0\omega_0^2\sin2\varkappa_0+3fM_2R_0^{-5}(A_1-A_2)\sin2(\alpha_{10}-\varkappa_0)(A_2-A_3)\sin2\theta_0+$$
$$+3fM_1R_0^{-5}(B_1-B_2)\sin2(\alpha_0-\varkappa_0)=0$$

Solutions (2.3)–(2.6) describe the rotation of the spheroid and gyrostat about their common center of mass G at the angular velocity $\omega_0=(K/S)_0$; in the case of solutions (2.3), (2.5), (2.6) the spheroid also rotates about its dynamic symmetry axis y_3 at the proper rotation velocity

$$\varphi^{\cdot} = K_{\varphi}/B_2 - \omega_0 \cos\alpha_0 \cos\beta_0$$

In the case of solution (2.4) $\varphi^{\cdot} = 0$. For solutions (2.3)–(2.5) the planes of motion of the centers of mass of the spheroid and gyrostat coincide and the axis of inertia x_3 of the gyrostat is directed along the line of centers of mass PC. The axis y_2 of proper rotation of the spheroid is perpendicular to the orbital plane for solution (2.3), lies along the line of centers PC for solution (2.4), and is perpendicular to the line of centers PC, forming the constant angle β_0 with the axis η_3 for solution (2.5).

For solution (2.6) the line PC of the centers of mass of the gyrostat and spheroid forms the constant angle $\varkappa_0$ with the orbital planes of the centers of mass of the spheroid and gyrostat which are parallel (the distance between them is equal to $R_0 \sin\varkappa_0$ and is on the order of l^2/R); the principal axis of inertia x_1 of the gyrostat is directed along the velocity vector of its center of mass, and the quantity θ_0 is equal to the angle between the axes η_3 and x_2. Such motions in the case of a gyrostatic moving in a central Newtonian force field were first obtained by Stepanov [2] and Roberson [3, 4].

3. The sufficient conditions of stability of the above steady motions of a spheroid and gyrostat are obtainable as the Sylvester conditions of positive definiteness of the second variation of the function W_1.

It is easy to verify that of the conditions of stability of the steady motions of the spheroid + gyrostat system with respect to the variables

$$R,\ \varkappa,\ \alpha,\ \beta,\ \beta_1,\ \beta_2,\ \beta_3,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ R^{\cdot},\ \varkappa^{\cdot},\ \sigma^{\cdot},\ \alpha^{\cdot},\ \beta^{\cdot},\ \varphi^{\cdot},\ \beta_1^{\cdot},\ \beta_2^{\cdot},\ \beta_3^{\cdot},\ \gamma_1^{\cdot},\ \gamma_2^{\cdot},\ \gamma_3^{\cdot}$$

the conditions $(\partial^2 W_1/\partial\varkappa^2)_0 > 0$, $(\partial^2 W_1/\partial R^2)_0 > 0$ are always fulfilled if the dimensions of the bodies are much smaller than the distance between their centers of mass; the remaining stability conditions are reducible to the following forms.

For solution (2.3),

$$(B_2 - B_1)\,\omega_0 + B_2\varphi^{\cdot} > 0, \quad \omega_0 (B_2 - B_1)\frac{M_2 + 3M_0}{M_2} + B_2\varphi^{\cdot} > 0 \tag{3.1}$$

$$A_1 > A_3, \quad A_2 > A_3, \quad A_2 + \frac{k_2}{\omega_0 \cos\theta_0} > A_3, \quad A_2 + \frac{k_2}{\omega_0 \cos^3\theta_0} > A_1 + \delta$$

for solution (2.4),

$$B_1 > B_2, \qquad A_1 > A_3, \qquad A_2 > A_3 \tag{3.2}$$

$$A_2 + \frac{k_2}{\omega_0 \cos\theta_0} > A_3, \qquad A_2 + \frac{k_2}{\omega_0 \cos^3\theta_0} > A_1 + \delta$$

for solution (2.5)

$$B_2 > B_1, \qquad A_1 > A_3, \qquad A_2 > A_3$$

$$A_2 + \frac{k_2}{\omega_0 \cos\theta_0} > A_3, \qquad A_2 + \frac{k_2}{\omega_0 \cos^3\theta_0} > A_1 + \delta \tag{3.3}$$

for solution (2.6)

$$B_1 > B_2, \quad A_1 - A_2 \sin^2\theta_0 - A_3 \cos^2\theta_0 > 0, \quad (A_2 - A_3)(1 - \mathrm{tg}^2\theta_0) > 0$$

$$(A_2 - A_3)\left(1 + \frac{3M_0}{M_1}\,\mathrm{tg}^2\theta_0\right) + \frac{k_2}{\omega_0 \cos^3\theta_0} > 0 \tag{3.4}$$

$$(A_1 - A_2 \sin^2\theta_0 - A_3 \cos^2\theta_0)\left(A_2 - A_1 + \frac{k_2}{\omega_0 \cos\theta_0}\right) + \frac{3M_0}{M_1}(A_1 - A_2)(A_2 - A_3)\sin^2\theta_0 > 0$$

where

$$\delta = \frac{3f\,(A_1 - A_2)^2 \sin^2\theta_0}{S_0 R_0^3 (\partial^2 W_1/\partial R^2)_0}\left\{M_1 M_2 - \frac{5}{2R_0^3}\left[(2A_3 - A_1 - A_2)\,M_2 + (B_1 - B_2)\,M_1\right]\right\}$$

In analyzing the above sufficient conditions (3.1)–(3.4) of stability of steady motions

(2.3)–(2.6) of the spheroid + gyrostat system, we see that each group of sufficient conditions of stability consists of two groups; one group contains the moments of inertia of the spheroid alone, the other the moments of inertia of the gyrostat alone. Each of these groups constitutes stability conditions similar to those obtained by Rumiantsev [5] for the corresponding motions of a symmetric satellite and gyrostat satellite about a fixed attracting center. In our case the role of the attracting center is played by the center of mass G of the system.

4. Let us consider the case of a dynamically symmetric gyrostat when $A_1 = A_3 \neq A_2$ and $k_1 = k_3 = 0$, $k_2 = \text{const}$.

In addition to integrals (2.1) and (2.2) our system also has the integral

$$\partial L / \partial \varphi_1^{\cdot} = A_2 (\sigma^{\cdot} \cos \alpha_1 \cos \beta_1 + \varphi_1^{\cdot} + \alpha_1^{\cdot} \sin \beta_1) + k_2 = K\varphi_1 \tag{4.1}$$

which expresses the constancy of the projection of the moment of momenta of the gyrostat on its dynamic symmetry axis x_2.

Ignoring the cyclical coordinates σ, φ, φ_1, we obtain the following expression for the altered potential energy U of the system:

$$U(R, \varkappa, \alpha, \beta, \alpha_1, \beta_1) = \frac{K_1^2}{2S_1} + \frac{K_\varphi^2}{2B_2} + \frac{(K_{\varphi_1} - k_2)^2}{2A_2} + \Pi$$

where

$$\Pi = 3f \frac{M_2}{2R^3} \Big\{A_1 [\sin^2(\alpha_1 - \varkappa) \sin^2 \beta_1 + \cos^2(\alpha_1 - \varkappa)] + A_2 \sin^2(\alpha_1 - \varkappa) \cos^2 \beta_1 - \\ - \frac{2A_1 + A_2}{3}\Big\} + \frac{3fM_1}{2R^3} \Big\{B_1 [\sin^2(\alpha - \varkappa) \sin^2 \beta + \cos^2(\alpha - \varkappa)] + \\ + B_2 \sin^2(\alpha - \varkappa) \cos^2 \beta - \frac{2B_1 + B_2}{3}\Big\} - f\frac{M_1 M_2}{R}$$

$$S_1 = M_0 R^2 \cos^2 \varkappa + A_1 (\sin^2 \alpha_1 + \cos^2 \alpha_1 \sin^2 \beta_1) + B_1 (\sin^2 \alpha + \cos^2 \alpha \sin^2 \beta)$$

$$K_1 = K_\sigma - K_\varphi \cos \alpha \cos_\beta - K_{\varphi_1} \cos \alpha_1 \cos \beta_1$$

The steady motions of the system can be determined from the equation

$$\delta U = 0$$

This equation has the following solutions:

$$R = R_0, \ \varkappa = 0, \ \alpha = 0, \ \beta = 0, \ \alpha_1 = 0, \ \beta_1 = 0 \tag{4.2}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 0, \ \cos \beta = K_\varphi / \omega_0 B_1, \ \alpha_1 = 0, \ \cos \beta_1 = K_{\varphi_1} / \omega_0 A_1 \tag{4.3}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 0, \ \beta = 0, \ \alpha_1 = 0, \ \cos \beta_1 = K_{\varphi_1} / \omega_0 A_1 \tag{4.4}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 0, \ \cos \beta = K_\varphi / \omega_0 B_1, \ \alpha_1 = 0, \ \beta_1 = 0 \tag{4.5}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 0, \ \beta = 0, \ \alpha_1 = 1/2\,\pi, \ \beta_1 = 0, \ K_{\varphi_1} = 0 \tag{4.6}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 1/2\,\pi, \ \beta = 0, \ K_\varphi = 0, \ \alpha_1 = 0, \ \beta_1 = 0 \tag{4.7}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 0, \ \cos \beta = K_\varphi / \omega_0 B_1, \ \alpha_1 = 1/2\,\pi, \ \beta_1 = 0, \ K_{\varphi_1} = 0 \tag{4.8}$$

$$R = R_0, \ \varkappa = 0, \ \alpha = 1/2\,\pi, \ \beta = 0, \ K_\varphi = 0, \ \alpha_1 = 0, \ \cos \beta_1 = K_{\varphi_1} / \omega_0 A_1 \tag{4.9}$$

$$\begin{gathered} R = R_0, \ \varkappa = \varkappa_0, \ \cos \alpha = \cos \alpha_0 = \frac{\omega_0 K_\varphi}{\omega_0^2 B_1 - 3fM_1 R_0^{-3} (B_2 - B_1)} + \delta_2, \ \beta = 0 \\ \cos \alpha_{10} = \frac{\omega_0 K_{\varphi_1}}{\omega_0^2 A_1 - 3fM_2 R_0^{-3} (A_2 - A_1)} + \delta_3, \ \beta_1 = 0 \end{gathered} \tag{4.10}$$

Solutions (4.2)–(4.9) exist under the condition

$$M_0\omega_0^2R_0^3 = f\,\{M_1M_2 - {}^3/_2\, R_0^{-2}\,[(A_2 - A_1)\,M_2 + (B_2 - B_1)\,M_1]\}$$

and solution (4.10) under the condition

$$M_0\omega_0^2R_0^3\cos^2\varkappa_0 = f\,\Big\{M_1M_2 - \frac{9M_2}{2R_0^2}\Big[A_1\cos^2(\alpha_{10}-\varkappa_0) + A_2\sin^2(\alpha_{10}-\varkappa_0) - \frac{2A_1+A_2}{3}\Big] -$$

$$- \frac{9M_1}{2R_0^2}\Big[B_1\cos^2(\alpha_0-\varkappa_0) + B_2\sin^2(\alpha_0-\varkappa_0) - \frac{2B_1+B_2}{3}\Big]\Big\}$$

$$M_0\omega_0^2R_0^5\sin 2\varkappa_0 + 3f\,M_2\,(A_1 - A_2)\sin 2\,(\alpha_{10}-\varkappa_0) +$$
$$+ 3fM_1\,(B_1 - B_2)\sin 2\,(\alpha_0 - \varkappa_0) = 0$$

where $\varkappa_0$, δ_2 and δ_3 are quantities on the order of l^2 / R^2.

These solutions describe the steady motions of the spheroid and dynamically symmetric gyrostat about their common center of mass G at the angular velocity $\omega_0 = (K_1/S_1)_0$, with the spheroid rotating about its axis of symmetry y_2 at the proper rotation velocity

$$\varphi^{\cdot} = K_\varphi/B_2 - \omega_0\cos\alpha\cos\beta$$

and the dynamically symmetric gyrostat rotating about its axis of symmetry x_2 at the proper rotation velocity $\varphi_1^{\cdot} = K_{\varphi_1}/A_2 - \omega_0\cos\alpha_1\cos\beta_1$

In solutions (4.2)–(4.9) the planes of motion of the centers of mass of the spheroid and dynamically symmetric gyrostat coincide. In solution (4.10) these orbital planes are parallel, lying at the distance $R_0\sin\varkappa_0$ on the order of l^2/R from each other.

For solution (4.2) the axes of proper rotation of the spheroid y_2 and symmetric gyrostat x_2 are perpendicular to the orbital plane. This solution was first obtained by Kondurar' [6, 7].

For solution (4.3) the axes of proper rotation of the spheroid y_2 and symmetric gyrostat x_2 are perpendicular to the line of centers PC and form the constant angles β_0 and β_{10} respectively, with the axis η_2.

For solution (4.4) the axis of proper rotation y_2 of the spheroid is perpendicular to the orbital plane, and the axis of proper rotation x_2 of the symmetric gyrostat perpendicular to the line of centers PC, forming the constant angle β_{10} with the axis η_2.

For solution (4.5) the axis of proper rotation y_2 of the spheroid is perpendicular to the line of centers PC and forms the constant angle β_0 with the axis η_2; the axis of proper rotation x_2 of the symmetric gyrostat is perpendicular to the orbital plane.

For solution (4.6) the axis of proper rotation y_2 of the spheroid is perpendicular to the orbital plane, and the axis of proper rotation x_2 of the symmetric gyrostat is directed along the line of centers PC; in this case the gyrostat does not rotate about the axis x_2.

For solution (4.7) the axis of proper rotation y_2 of the spheroid is directed along the line of centers PC, but the spheroid does not rotate about the axis y_2; the axis of proper rotation x_2 of the symmetric gyrostat is perpendicular to the orbital plane.

For solution (4.8) the axis of proper rotation y_2 of the spheroid is perpendicular to the line of centers PC and forms the constant angle β_0 with the axis η_2; the axis of proper rotation x_2 of the symmetric gyrostat is directed along the line of centers, but the gyrostat does not rotate about the axis x_2.

For solution (4.9) the axis of proper rotation y_2 of the spheroid is directed along the line of centers, but the center does not rotate about the axis y_2; the axis of proper rotation x_2 of the symmetric gyrostat is perpendicular to the line of centers and forms the constant angle β_{10} with the axis η_2.

For solution (4.10) the line of centers PC forms the constant angle $\varkappa_0$ with the orbital planes of the centers of mass of the spheroid and symmetric gyrostat; the axes of symmetry y_2 of the spheroid and x_2 of the gyrostat lie in the plane Q and form the constant angles α_0 and α_{10}, respectively, with the axis η_2.

5. The sufficient conditions of stability of the above steady motions of a spheroid and symmetric gyrostat are obtainable as the Sylvester conditions of positive definiteness of the second variation of the function U. These conditions can be expressed as follows:

for solution (4.2),

$$K_\varphi > B_1\omega_0, \qquad K_\varphi - B_1\omega_0 + 3M_0\omega_0 M_2^{-1}(B_2 - B_1) > 0$$
$$K_{\varphi_1} > A_1\omega_0, \qquad K_{\varphi_1} - A_1\omega_0 + 3M_0\omega_0 M_1^{-1}(A_2 - A_1) > 0 \tag{5.1}$$

for solution (4.3),

$$B_2 > B_1, \qquad A_2 > A_1 \tag{5.2}$$

for solution (4.4),

$$K_\varphi > B_1\omega_0, \quad K_\varphi - \omega_0 B_1 + 3M_0\omega_0 M_2^{-1}(B_2 - B_1) > 0, \quad A_2 > A_1 \tag{5.3}$$

for solution (4.5),

$$B_2 > B_1, \quad K_{\varphi_1} > A_1\omega_0, \quad K_{\varphi_1} - A_1\omega_0 + 3M_0\omega_0 M_1^{-1}(A_2 - A_1) > 0 \tag{5.4}$$

for solution (4.6),

$$K_\varphi > B_1\omega_0, \quad K_\varphi - B_1\omega_0 + 3M_0\omega_0 M_2^{-1}(B_2 - B_1) > 0, \quad A_1 > A_2 \tag{5.5}$$

for solution (4.7),

$$B_1 > B_2, \quad K_{\varphi_1} > A_1\omega_0, \quad K_{\varphi_1} - A_1\omega_0 + 3M_0\omega_0 M_1^{-1}(A_2 - A_1) > 0 \tag{5.6}$$

for solution (4.8),

$$B_2 > B_1, \qquad A_1 > A_2 \tag{5.7}$$

for solution (4.9),

$$B_1 > B_2, \qquad A_1 > A_2 \tag{5.8}$$

for solution (4.10),

$$B_1 > B_2, \qquad A_1 > A_2 \tag{5.9}$$

Conditions (5.1)–(5.9) are the sufficient conditions of stability of steady motions (4.2)–(4.10) with respect to the variables

$$R, \varkappa, \alpha, \beta, \alpha_1, \beta_1, R^\cdot, \varkappa^\cdot, \sigma^\cdot, \alpha^\cdot, \beta^\cdot, \varphi^\cdot, \alpha_1^\cdot, \beta_1^\cdot, \varphi_1^\cdot$$

By virtue of Kelvin's theorem [8], steady motions (4.2)–(4.10) become unstable if we replace one of the inequalities in conditions (5.1)–(5.9) by one of opposite sign. Steady motions (4.4)–(4.7) are also unstable if we replace all three inequalities in conditions (5.3)–(5.6) by inequalities of opposite sign. Steady motion (4.2) is unstable if we replace any three inequalities of condition (5.1) by inequalities of opposite sign.

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ON THE CONSTRUCTION OF SOLUTIONS OF QUASILINEAR NONAUTONOMOUS SYSTEMS IN RESONANCE CASES

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We consider a system with n degrees of freedom, of the following form:

$$x_s^{\cdot} = -\lambda_s y_s + \mu X_{s1}(x, y, t) + \mu^2 X_{s2}(x, y, t) + \ldots + f_{s0}(t) + \mu f_{s1}(t) + \ldots \qquad (0.1)$$
$$y_s^{\cdot} = \lambda_s x_s + \mu Y_{s1}(x, y, t) + \mu^2 Y_{s2}(x, y, t) + \ldots + \varphi_{s0}(t) + \mu\varphi_{s1}(t) + \ldots$$
$$x \equiv (x_1, \ldots, x_n), \quad y \equiv (y_1, \ldots, y_n) \qquad (s = 1, \ldots, n)$$

Here $X_{s1}, \ldots,$ $Y_{s1}, \ldots$ are polynomials of an arbitrarily high degree in x and y with continuous coefficients which are 2π-periodic in t. The functions $f_{s0}, \ldots,$ $\varphi_{s0}, \ldots$ are continuous and have the same period. Quantity μ is a small parameter. We assume that both internal and external resonance are present in the system.

There exist various well worked out methods of investigating the oscillations of quasilinear nonautonomous systems in resonance cases (method of small parameter, method of averaging, e. a.). these reduce the problem of constructing the oscillations accurate to the first degree of the small parameter to obtaining solutions of, so called, fundamental (generating) amplitude equations. In the case of a system with several degrees of freedom, these equations represent a system of nonlinear algebraic equations, for which general solution does not esist. Thus, one problem leads to another which is no less complex.

In the present paper we use the results of [1, 2] to develop a method of constructing both periodic and almost-periodic solutions. This allows us to obtain the values of the fundamental amplitudes from a system of linear algebraic equations, when the order of the highest form accompanying μ is not greater than three. If X_{s1} and Y_{s1} contain terms of the order higher than three, then the equations defining the fundamental amplitudes will be also nonlinear, but simpler than those appearing in the method of small parameters, method of averaging, etc.